Improvement of Plate and Shell Finite Elements by Mixed Formulations

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Mixed formulations are introduced as a means for reducing severe constraints in finite-element derivations. For plate bending elements to include transverse shear effect and to be applicable also to thin plates, the method can reduce the conditions of constraints of zero transverse shear strain energy. For shell elements, the constraints of rigid-body modes can be lessened similarly. Certain mixed formulations have been shown to be equivalent to displacement models with reduced integration scheme. But the present approach provides a general and rational method for the finite-element development. Illustrative examples include plate, circular arch, and shell elements.

I. Introduction

NUMEROUS finite elements are available for linear and nonlinear analyses of plate and shell structures. A majority of elements employ the usual Kirchhoff assumption, which neglects the effect of the transverse shear deformation. The Kirchhoff assumption reduces the number of variables but introduces higher-order derivatives in the formulation of plate and shell elements. Thus, if a stiffness matrix is generated from the principle of minimum potential energy, the compatibility conditions for a plate bending element would include the difficult task of maintaining the continuity of the slope of displacement across the element boundaries.[1,2]

It is possible to relax the compatibility requirement by using alternative variational principles. The hybrid stress method of Pian[3,4] and the hybrid displacement method by Tong[5] offer an advantage over the conventional assumed displacement method in this context. Among various hybrid elements, we cite plate bending elements by Cook[6] and Allman,[7] and plate and shell elements by Kikuchi and Ando.[8] The mixed formulation by Herrmann[9] was used to relax the compatibility requirement, but it does not fit the ordinary format of the matrix displacement method in that, in addition to nodal displacements, nodal stresses are kept in the final assembled equations.

By abandoning the Kirchhoff assumption, the interelement compatibility requirement no longer poses a serious problem. In the principle of minimum potential energy for plate bending, the rotation angles appear as variables in addition to normal deflection. But it is known that the so-called thick-plate formulation does not give reliable solutions for thin-plate problems when an exact order of integration is used.[10,11] Therefore Wempner et al.[12] introduced the concept of the discrete Kirchhoff assumption in which the Kirchhoff assumption is imposed at selected points, and the transverse shear strain energy term is neglected. Batoz et al.[13] used this concept for their shell element. On the other hand, reduced integration by Zienckwicz et al.[10] and by Pawsey and Clough[11] utilizes a lower order of integration and has proved to be very successful. It is recognized now that the difficulty in using a thick-plate formulation for thin-plate bending problems lies on the existence of severe constraints because of the condition of zero transverse shear strain energy.

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In thin-shell analyses, in addition to the aforementioned problems, we must consider the ability of a finite element to represent constant strain states and rigid-body modes. Haisler and Stricklin[14] and Mebane and Stricklin[15] discuss this problem. Morris[16] and Gallagher[17] give interesting accounts pertinent to shell analysis in general.

This paper intends to clarify the nature of the constraint of these two problems. Some plate bending elements are to be derived by the Hellinger-Reissner principle and a modified Hellinger-Reissner principle. Also, the Hellinger-Reissner principle is to be used for the derivation of a shell element by a mixed formulation. The resulting equations fit the format of the matrix displacement method.

II. Plate Bending Elements with Transverse Shear Effect

Simple Beam Example

We consider, for simplicity, the bending of a simple beam that has a unit width, a length L, and a depth t. In the conventional assumed displacement method, the element stiffness matrix of a beam with transverse shear effect is derived from the principle of minimum potential energy. In nondimensional form, for which the normal deflection w and the distance x along the beam are written as

$$x = Lx', \quad w = Lw' \tag{1}$$

the functional π_p is

$$\pi_p = \frac{1}{2}\frac{Et^3}{L}\left[\int_0^1 \frac{1}{12}\left(\frac{d\phi}{dx'}\right)^2 dx' + \frac{G\beta}{E}\left(\frac{L}{t}\right)^2 \int_0^1 \left(\phi + \frac{dw'}{dx'}\right)^2 dx'\right] - W \tag{2}$$

where

ϕ = rotation
β = form factor (= 5/6 for a rectangular cross section)
$-W$ = potential energy due to applied loads
E = Young's modulus
G = shear modulus

The first and second integrals in Eq. (2) represent the bending energy and the transverse shear strain energy, respectively. As the beam becomes thinner and thinner, the magnitude of the transverse shear strain energy relative to the bending energy becomes smaller. Thus a finite element must be able to represent a small or zero transverse shear strain

Table 1 Maximum nondimensional displacement $\bar{w}=EIw/pL^3$ for a cantilever beam under tip load p

Length/thickness	10	100
Exact solution	0.3353	0.3333
5 RB4 elements	0.3320	0.3300
20 displacement DB4 elements	0.3037	0.2920×10^{-1}

energy as accurately as possible. Otherwise, any error resulting from inaccurate representation of the transverse shear strain energy will be magnified by $(L/t)^2$ and dominate the bending energy. To be able to represent the state of zero transverse shear strain energy, it is necessary that

$$\phi+\frac{dw'}{dx'}=0 \tag{3}$$

Equation (3) imposes severe constraints on the assumed deformation mode of a finite element. Consider a four-degree-of-freedom (DOF) element designated as DB4 with

$$\phi=a_1+a_2x', \quad w'=a_3+a_4x' \tag{4}$$

then, from Eq. (3),

$$a_1+a_4=0, \quad a_2=0 \tag{5}$$

Therefore, two constraints are present among four degrees of freedom. In actuality, the number of constraints must be measured on an entire structural level. But it is helpful to count the number of constraints on an element level in order to determine relative efficiency of various elements with the same number of degrees of freedom. To demonstrate the effect of the constraints, a cantilever beam under tip load p was analyzed with 20 elements. The calculated nondimensional maximum displacements $\bar{w}$ are given in Table 1. It is seen that results by the DB4 element are very sensitive to the thickness ratios.

Mixed Formulations

We consider next a mixed formulation using the Hellinger-Reissner principle,[18] with a functional given in terms of independent stresses σ_{ij} and displacements u_i as follows:

$$\pi_R=\int_v \{\sigma_{ij}\bar{e}_{ij}-\tfrac{1}{2}S_{ijk\ell}\sigma_{ij}\sigma_{k\ell}\}dv-\int_{S_\sigma}\bar{T}_iu_i ds \tag{6}$$

in which

$\bar{e}_{ij}$ $=\frac{1}{2}(u_{i,j}+u_{j,i})$
$S_{ijk\ell}$ = compliance tensor
$\bar{T}_i$ = applied tractions

The strains e_{ij} are related to the stresses by

$$e_{ij}=S_{ijk\ell}\sigma_{k\ell} \tag{7a}$$

or

$$\sigma_{ij}=C_{ijk\ell}e_{k\ell} \tag{7b}$$

Introducing Eq. (7b) into Eq. (6), we obtain

$$\pi_R=\int_v (C_{ijk\ell}e_{k\ell}\bar{e}_{ij}-\tfrac{1}{2}C_{ijk\ell}e_{ij}e_{k\ell})dv-\int_{S_\sigma}\bar{T}_iu_i ds \tag{8}$$

The Hellinger-Reissner functional expressed in Eq. (8) appears to be more convenient for thin structures when material nonlinearities such as creep or plasticity effects are included in the analysis. Stresses across the thickness could be highly nonlinear or piecewise continuous in these cases, whereas strains may be considered linear. It means that, if a mixed element is to be formulated using Eq. (6), it is necessary to assume a higher-order stress field in these cases.

For plate bending problems, the functional in Eq. (8) is expressed in matrix form as follows:

$$\pi_R=\int(\kappa^TC_\kappa\bar{\kappa}-\tfrac{1}{2}\kappa^TC_\kappa\kappa)dxdy+\int(\gamma^TC_\gamma\bar{\gamma}-\tfrac{1}{2}\gamma^TC_\gamma\gamma)dxdy-W \tag{9}$$

in which

$$\kappa=\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}=\text{curvature strains}$$

$$\gamma=\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\text{transverse shear strains}$$

$$\bar{\kappa}=\begin{Bmatrix}\dfrac{\partial\phi}{\partial x}\\ \dfrac{\partial\psi}{\partial y}\\ \dfrac{\partial\phi}{\partial y}+\dfrac{\partial\psi}{\partial x}\end{Bmatrix}, \quad \bar{\gamma}=\begin{Bmatrix}\phi+\dfrac{\partial w}{\partial x}\\ \psi+\dfrac{\partial w}{\partial y}\end{Bmatrix}$$

with ϕ, ψ = rotation angles, and C_κ, C_γ are appropriate elastic constant matrices. A mixed element is derived by assuming curvature strains and transverse shear strains to be polynominals in x and y with unknown parameters β_i and α_i, respectively, which are independent for the element. Rotation angles and normal displacement are assumed in terms of nodal values q_i. Thus,

$$\kappa=P_\beta\beta, \quad \gamma=P_\alpha\alpha \tag{10a}$$

$$u=\begin{Bmatrix}\phi\\ \psi\\ w\end{Bmatrix}=Aq \tag{10b}$$

Then

$$\bar{\kappa}=B_\kappa q, \quad \bar{\gamma}=B_\gamma q \tag{11}$$

Substituting these into Eq. (8),

$$\pi_R=\sum_n\left(\beta^TG_\beta q-\frac{1}{2}\beta^TH_\beta\beta+\alpha^TG_\alpha q-\frac{1}{2}\alpha^TH_\alpha\alpha-q^TQ\right) \tag{12}$$

with

G_β $=\int P_\beta^TC_\kappa B_\kappa\, dxdy$
H_β $=\int P_\beta^TC_\kappa P_\beta\, dxdy$
G_α $=\int P_\alpha^TC_\gamma B_\gamma\, dxdy$
H_α $=\int P_\alpha^TC_\gamma P_\alpha\, dxdy$
q^TQ $=W$
n = number of elements

By taking $\delta\pi_R=0$ with respect to β and α, we can express β and α in terms of q:

$$\beta=H_\beta^{-1}G_\beta q, \quad \alpha=H_\alpha^{-1}G_\alpha q \tag{13}$$

Substituting Eq. (13) into Eq. (12),

$$\pi_R = \sum_n \left(\frac{1}{2} q^T k q - q^T Q \right)$$

with

$$k = G_\beta^T H_\beta^{-1} G_\beta + G_\alpha^T H_\alpha^{-1} G_\alpha \tag{14}$$

For a thin plate, the constraints on this mixed element are obtained by setting the transverse shear strain energy term to be zero, i.e.,

$$\tfrac{1}{2} q^T G_\alpha^T H_\alpha^{-1} G_\alpha q = 0 \tag{15}$$

or

$$G_\alpha q = 0 \tag{16}$$

It is seen that the maximum possible number of constraints for an element is the same as the rank of the G_α matrix. Therefore, it is possible to control the number of constraints by choosing an appropriate number of α_i.

A part of the Euler equations of the Hellinger-Reissner principle is the following curvature-displacement relation and the relation between transverse shear strains and displacements:

$$\kappa = \bar{\kappa} \tag{17}$$

$$\gamma = \bar{\gamma} \tag{18}$$

If the curvature-displacement relation in Eq. (17) is satisfied exactly in Eq. (9), the modified Hellinger-Reissner principle is derived as follows:

$$\pi_{mR} = \tfrac{1}{2} \int \bar{\kappa}^T C_\kappa \bar{\kappa} \mathrm{d}x\mathrm{d}y + \int (\gamma^T C_\gamma \bar{\gamma} - \tfrac{1}{2} \gamma^T C_\gamma \gamma) \mathrm{d}x\mathrm{d}y - W \tag{19}$$

The curvature strains do not appear as variables in the preceding expression. The procedure for a mixed formulation is similar to the case of the Hellinger-Reissner principle. That is, we assume

$$\gamma = P_\alpha \alpha \tag{20a}$$

$$u = Aq \tag{20b}$$

or

$$\bar{\kappa} = Bq \tag{20c}$$

Then,

$$\pi_{mR} = \sum_n \left(\frac{1}{2} q^T k_\beta q + \alpha^T G_\alpha q - \frac{1}{2} \alpha^T H_\alpha \alpha - q^T Q \right) \tag{21}$$

with

$$k_\beta = \int B^T C_\kappa B \mathrm{d}x\mathrm{d}y$$

By taking $\delta \pi_{mR} = 0$ with respect to α, we obtain

$$\alpha = H_\alpha^{-1} G_\alpha q \tag{22}$$

Substituting α from Eq. (22) into Eq. (21),

$$\pi_{mR} = \sum_n \left(\frac{1}{2} q^T k_\beta q + \frac{1}{2} q^T G_\alpha^T H_\alpha^{-1} G_\alpha q - q^T Q \right) \tag{23}$$

By setting the transverse shear strain energy equal to zero, we obtain the same constraint as that in Eq. (16).

We also observe that, if the transverse shear strain-displacement relation in Eq. (18) is satisfied exactly, the functional for the principle of minimum potential energy is obtained. As an example, the same cantilever beam problem was solved by using a mixed element. The element has four DOF's with linear w and ϕ as in Eq. (4), and the transverse shear strain is assumed to be constant, resulting in one constraint for the element. The displacement element has two constraints, as mentioned before. If a constant curvature is assumed in the Hellinger-Reissner principle, the resulting element will be the same as that from the modified Hellinger-Reissner principle. We designate the element as RB4. The nondimensional maximum displacement is given in Table 1 in comparison with the results by the displacement elements. Note that only five mixed elements were used, while twenty elements were used for the displacement method. It is seen that the mixed element is not sensitive to the thickness ratios and gives reliable solutions.

Some Plate Bending Elements

We describe here some plate bending elements derived by the formulation discussed previously and give solutions for a numerical example problem. Three eight-node curved quadrilateral elements with 24 DOF's and three six-node curved triangular elements with 18 DOF's were derived. For description of geometry and displacement, isoparametric representation is utilized. Each component of curvature strains and/or transverse shear strains is assumed to have the same distribution in x and y. These elements are listed in Table 2. It is to be noted that elements R18 and MR18 are identical for a geometry with straight sides and midside nodes. These elements were used to analyze a simply supported square plate under a constant pressure for various thickness ratios. Because of symmetry, only a quarter of the plate was discretized by 2×2 rectangular meshes. For six-node triangular elements, a 1×1 mesh is made of four elements. Table 3 lists the nondimensional maximum deflection.

It is seen that among the eight-node elements the MR24A element gives good results for the entire range of thickness ratios considered. On the other hand, the R24 and MR24 elements tend to be too stiff for very thin plates. All six-node elements give reliable solutions in general. For very thin plates, the MR18A element, with seven integration points, appears to yield the most accurate solutions.

III. Shell Elements

Circular Arch as a Simple Shell

A circular arch is an important structural member in itself, and furthermore it exhibits essential characteristics of shell structures. Consider the functional for the principle of minimum potential energy for a circular arch of a unit width and radius R written as

$$\pi_P = \frac{1}{2} \int \left\{ Et \left(\frac{\mathrm{d}u}{\mathrm{d}S} + \frac{w}{R} \right)^2 + \frac{1}{12} Et^3 \left(\frac{1}{R} \frac{\mathrm{d}u}{\mathrm{d}S} - \frac{\mathrm{d}^2 w}{\mathrm{d}S^2} \right)^2 \right\} \mathrm{d}S - W \tag{24}$$

where

u = in-plane displacement
w = normal displacement
S = coordinate along the arch

If we introduce an angle ϕ such that $\mathrm{d}S = R\mathrm{d}\phi$, then

$$\pi_P = \frac{E}{2} \left(\frac{t}{R} \right)^3 \int \left\{ \left(\frac{R}{t} \right)^2 \left(\frac{\mathrm{d}u}{\mathrm{d}\phi} + w \right)^2 + \frac{1}{12} \left(\frac{\mathrm{d}u}{\mathrm{d}\phi} - \frac{\mathrm{d}^2 w}{\mathrm{d}\phi^2} \right)^2 \right\} \mathrm{d}\phi - W \tag{25}$$

Table 2 List of plate bending elements by mixed formulations

No. of nodes (DOF's)	8 (24)			6 (18)		
Element name	R24	MR24	MR24A	R18	MR18	MR18A
Variational principle	Hellinger-Reissner	Modified Hellinger-Reissner	Modified Hellinger-Reissner	Hellinger-Reissner	Modified Hellinger-Reissner	Modified Hellinger-Reissner
Assumed Curvature strains	$1,x,y,xy$	...	...	$1,x,y$	...	...
Assumed transverse shear strains	$1,x,y,xy$	$1,x,y,xy$	$1,x,y$	$1,x,y$	$1,x,y$	$1,x+y$
No. of α_i's	8	8	6	6	6	4

Table 3 Maximum nondimensional normal deflection $\bar{w}=100\,Dw/pa^4$ for a simply supported plate under uniform pressure p [a]

Element name		R24	MR24	MR24A	R18 and MR18		MR18A		Exact solution
Integration points		2×2 (exact)	3×3 (exact)	3×3 (exact)	7	3	7	3	
	10	0.4270	0.4269	0.4273	0.4272	0.4257	0.4347	0.4330	0.4273
	20	0.4111	0.4110	0.4113	0.4105	0.4079	0.4143	0.4121	0.4115
	50	0.4059	0.4057	0.4069	0.4049	0.4019	0.4077	0.4051	0.4071
a/t	100	0.4020	0.4019	0.4062	0.4040	0.4010	0.4068	0.4038	0.4064
	300	0.3655	0.3654	0.4061	0.4042	0.4008	0.4068	0.4034	0.4062
	1000	0.1868	0.1867	0.4060	0.4044	0.4008	0.4068	0.4034	0.4062

[a] a = length of side, t = thickness, D = bending rigidity.

In the preceding expression, the first integral and the second integral represent the stretching energy and the bending energy, respectively. If displacements u and w are assumed to have a polynomial distribution along ϕ, rigid-body modes are not included explicitly in the assumed displacement modes. We focus our attention on the stretching part, where any error due to inaccurate representation of rigid-body modes or constant strain states is to be magnified by the factor $(R/t)^2$. To be able to represent a rigid-body mode in the stretching part, it is necessary that

$$\frac{\mathrm{d}u}{\mathrm{d}\phi}+w=0 \tag{26}$$

pointwise. Equation (26) imposes constraints on the deformation mode of a finite element derived from the principle of minimum potential energy. For example, consider an arch element with linear u and cubic w, i.e.,

$$u=a_1+a_2\phi \tag{27a}$$

$$w=a_3+a_4\phi+a_5\phi^2+a_6\phi^3 \tag{27b}$$

Then, according to Eq. (26),

$$a_2+a_3=0, \quad a_4=a_5=a_6=0 \tag{28}$$

Therefore, among six DOF's there exist four constraints that will limit severely the performance of the element.

As in the case of the plate bending problem, we utilize the Hellinger-Reissner principle, whose functional is given as

$$\pi_R=\int\left\{Et\epsilon\left(\frac{\mathrm{d}u}{\mathrm{d}S}+\frac{w}{R}\right)-\frac{1}{2}Et\epsilon^2\right\}\mathrm{d}S$$
$$+\int\left\{\frac{1}{12}Et^3\kappa\left(\frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}S}-\frac{\mathrm{d}^2w}{\mathrm{d}S^2}\right)\right.$$
$$\left.-\frac{1}{2}\frac{1}{12}Et^3\kappa^2\right\}\mathrm{d}S-W \tag{29}$$

Here the in-plane strain ϵ and curvature κ are introduced as additional variables. As usaal, a mixed element is formulated by assuming u and w in terms of nodal displacements q_i and ϵ and κ as polynomials in ϕ with unknown parameters α_i and β_i, respectively, i.e.,

$$\epsilon=\boldsymbol{P}_\alpha\boldsymbol{\alpha}, \quad \kappa=\boldsymbol{P}_\beta\boldsymbol{\beta}, \quad \boldsymbol{u}=\left\{\begin{matrix}u\\w\end{matrix}\right\}=\boldsymbol{A}\boldsymbol{q} \tag{30}$$

Substituting these into Eq. (29), π_R is written as a function of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{q}$:

$$\pi_R=\sum_n\left(\boldsymbol{\alpha}^T\boldsymbol{G}_\alpha\boldsymbol{q}-\frac{1}{2}\boldsymbol{\alpha}^T\boldsymbol{H}_\alpha\boldsymbol{\alpha}+\boldsymbol{\beta}^T\boldsymbol{G}_\beta\boldsymbol{q}-\frac{1}{2}\boldsymbol{\beta}^T\boldsymbol{H}_\beta\boldsymbol{\beta}-\boldsymbol{q}^T\boldsymbol{Q}\right) \tag{31}$$

where $\boldsymbol{G}_\alpha$, $\boldsymbol{H}_\alpha$, $\boldsymbol{G}_\beta$, $\boldsymbol{H}_\beta$ are appropriately defined matrices. Taking $\delta\pi_R=0$ with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we obtain $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in terms of $\boldsymbol{q}$, and thus π_R can be written in terms of $\boldsymbol{q}$ only as follows:

$$\pi_R=\sum_n\left(\frac{1}{2}\boldsymbol{q}^T\boldsymbol{G}_\beta^T\boldsymbol{H}_\beta^{-1}\boldsymbol{G}_\beta\boldsymbol{q}+\frac{1}{2}\boldsymbol{q}^T\boldsymbol{G}_\alpha^T\boldsymbol{H}_\alpha^{-1}\boldsymbol{G}_\alpha\boldsymbol{q}-\boldsymbol{q}^T\boldsymbol{Q}\right) \tag{32}$$

in which the second part represents the stretching energy. Thus, by setting it to be zero, we obtain the constraint equation:

$$\boldsymbol{G}_\alpha\boldsymbol{q}=0 \tag{33}$$

Here the number of constraints is equal to the number of unknown parameters α_i.

By introducing the curvature-displacement relation

$$\kappa=\frac{1}{12}Et^3\left(\frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}S}-\frac{\mathrm{d}^2w}{\mathrm{d}S^2}\right) \tag{34}$$

into the Hellinger-Reissner principle, we obtain a modified

Hellinger-Reissner principle, written as

$$\pi_{mR} = \int \left\{ Et\epsilon \left(\frac{du}{dS} + \frac{w}{R} \right) - \frac{1}{2} Et\epsilon^2 \right\} dS + \frac{1}{2} \int \frac{1}{12} Et^3 \left(\frac{1}{R} \frac{du}{dS} - \frac{d^2 w}{dS^2} \right)^2 dS - W \tag{35}$$

The curvature strain κ does not appear as a variable. A mixed element is formulated in the usual manner, and the constraint on the element is the same as that in Eq. (33).

Numerical Examples

Several arch elements are listed in Table 4. Mixed elements were derived by the modified Hellinger-Reissner principle. But it is to be noted that the same element can be derived by the Hellinger-Reissner principle with properly assumed curvature strain. Using these elements, a circular ring of radius R pinched at two opposite points was analyzed. Because of symmetry, only a quarter of the ring was discretized for finite-element analysis. Results are given in Table 5 and Fig. 1. Table 5 gives maximum nondimensional deflection, and Fig. 1 shows nondimensional in-plane stress, which is defined as the in-plane force divided by the load p. The stresses calculated by the displacement element DA8 show severe fluctuations. This fluctuation grows with increasing R/t ratio. The result of the DA6 element shows similar but even more fluctuations. These results by displacement elements agree with those by Dawe,[19] who tested circular arch elements with various displacement assumptions. We observe that the present mixed elements are not sensitive to R/t ratios and give reliable results.

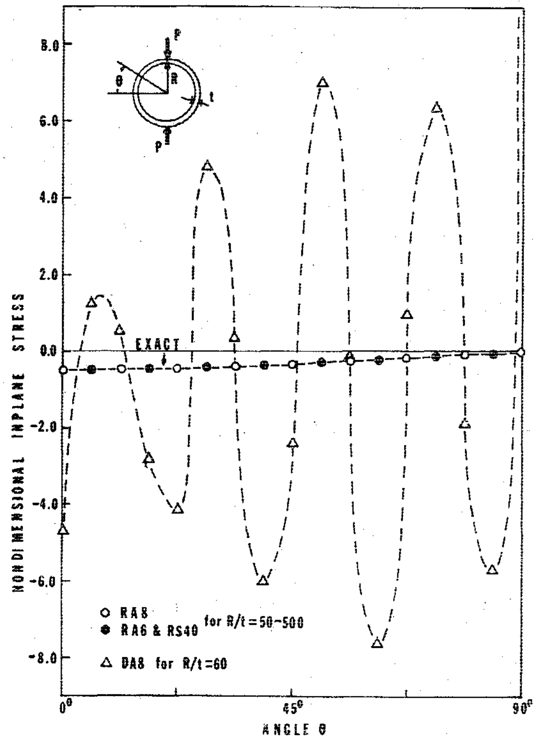

Fig. 1 Nondimensional in-plane stress for a pinched ring.

Shell Element by Mixed Formulation

Based on what we have learned in the previous discussion, we can derive shell elements. Many existing shell elements are derived from a general shell theory such as the Koiter-Sanders theory. But in this paper the "degenerate" shell element concept[10,11,20] is used because of its simplicity.

A corresponding shell element is formulated by the Hellinger-Reissner principle with the functional of the same form as that in Eq. (8). But now C_{ijkl}, e_{ij}, and $\bar{e}_{ij}$ are defined with respect to a local Cartesian coordinates with one coordinate normal to a shell surface and the other two coordinates embedded in the shell midsurface. The strain $\bar{e}_{ij}$ is obtained by a tensor transformation:

$$\bar{e}_{ij} = \ell_{iK} \ell_{jL} \bar{E}_{KL} \tag{36}$$

where derivatives of displacements U_k in $\bar{E}_{KL} = \frac{1}{2}(U_{K,L} + U_{L,K})$ are taken with respect to the global Cartesian coordinates, and ℓ_{iK} are the direction cosines. This transformation is required at each Gaussian integration point.

For the eight-node curved quadrilateral element (designated as RS40), the unknown strains are assumed as

$$e_x = \beta_1 + \beta_2 \xi + \beta_3 \eta + \beta_4 \xi\eta + \zeta(\beta_5 + \beta_6 \xi + \beta_7 \eta + \beta_8 \xi\eta) \tag{37a}$$

$$e_y = \beta_9 + \beta_{10} \xi + \beta_{11} \eta + \beta_{12} \xi\eta + \zeta(\beta_{13} + \beta_{14} \xi + \beta_{15} \eta + \beta_{16} \xi\eta) \tag{37b}$$

Table 4 List of arch elements

Element name	RA6	RA8	DA6	DA8
No. of DOF's	6	8	6	8
Variational principles	Modified Hellinger-Reissner	Modified Hellinger-Reissner	Minimum potential energy	Minimum potential energy
Assumed u	Linear	Cubic	Linear	Cubic
Assumed w	Cubic	Cubic	Cubic	Cubic
Assumed ϵ	Constant	Linear	...	...

Table 5 Maximum nondimensional normal deflection $\bar{w} = w / [(p/E)(R/t)^3]$ for a pinched ring

Element name	No. of elements	$R/t = 60$	$R/t = 100$	$R/t = 500$
RA6	8	0.8852	0.8852	0.8852
RA8	4	0.8925	0.8924	0.8924
DA6	8	0.1989×10^{-1}	0.3681×10^{-2}	0.4931×10^{-3}
DA8	4	0.8705	0.8322	0.8053
RS40[a]	8	...	0.8985	0.8984
Exact solution		0.8928	0.8927	0.8927

[a] Shell element.

$$e_{xy}=\beta_{17}+\beta_{18}\xi+\beta_{19}\eta+\beta_{20}\xi\eta+\zeta(\beta_{21}+\beta_{22}\xi+\beta_{23}\eta+\beta_{24}\xi\eta) \tag{37c}$$

$$\gamma_{xz}=\alpha_1+\alpha_2\xi+\alpha_3\eta \tag{37d}$$

$$\gamma_{yz}=\alpha_4+\alpha_5\xi+\alpha_6\eta \tag{37e}$$

where ξ, η, and ζ are the parent coordinates in the isoparametric transformation, and ζ is normal to the shell midsurface. It is to be noted that x, y, and z are components of the local Cartesian coordinates, with the z axis normal to the shell midsurface. Symbolically, Eqs. (36) and (37) are written as

$$\bar{e}=\boldsymbol{B}\boldsymbol{q} \tag{38a}$$

$$\boldsymbol{e}=\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix}=\boldsymbol{P}_\beta\boldsymbol{\beta} \tag{38b}$$

$$\boldsymbol{\gamma}=\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}=\boldsymbol{P}_\alpha\boldsymbol{\alpha} \tag{38c}$$

With these assumptions, the element stiffness matrix is symbolically of the same form as Eq. (14). The RS40 element has been used to solve the same pinched ring problem discussed earlier. Results obtained by using eight elements are given in Table 5 and Fig. 1. It is observed that solutions by four RA8 elements are slightly better than those by the present element. It indicates that, in spite of its simplicity in formulation, the degenerate element may not be considered the best among the elements with comparable degrees of freedom.

IV. Reduced Integration in Finite-Element Formulation

Reduced integration has been a successful scheme for plate bending and shell elements with transverse shear effects included. We shall discuss the relationship between the reduced integration scheme and the mixed formulation developed so far. First consider again the four-DOF beam element with linear w and ϕ. For the assumed displacement model DB4 element, two-point integration is needed for exact integration of the stiffness matrix. But, if one-point integration is used, it can be shown that the resulting stiffness matrix is the same as that of the element RB4 with constant transverse shear strain. Also, the stiffness matrix of the plate bending element R24 is the same as that of the eight-node reduced integration element with 2×2 Gaussian points. The same relation holds between the MR24 element and the reduced integration element, when 2×2 integration is used for the integration of the transverse shear strain energy. Also, by using the Hellinger-Reissner principle in Eq. (8), we can derive two-dimensional or three-dimensional elements. For an eight-node plane element, the reduced integration element with 2×2 Gaussian points gives the same stiffness matrix as that of a mixed element with strains assumed to be bilinear in the Cartesian coordinates x and y. Equivalence between the two formulations can be proved from Eq. (8) by establishing $e_{ij}=\bar{e}_{ij}$ at Gaussian points. Finally, it appears that plate bending elements by mixed formulations such as MR24A and MR18A are not derivable by the reduced integration method.

V. Conclusion

Difficulties associated with plate bending elements with transverse shear effect and shell finite elements have been identified. Formulations based on the Hellinger-Reissner principle offer a greater flexibility in the formulation of a finite element. Reduced integration elements are found to be identical to the present mixed elements. Therefore, it is argued that in these cases stresses must be calculated in a manner consistent with mixed formulations.

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